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Low Frequency Solution of Three-Dimensional

Scattering Problems

by
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August 1965

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Scientific Report No. 2

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ABSTRACT

A deficiency is pointed out in Stevenson's method of reducing the solution of electromagnetic scattering problems to a succession of standard potential problems whose solutions determine terms in the low frequency expansion of the scattered field. An alternate approach is presented, for perfectly conducting scatterers, which not only removes the difficulty but also is simpler and more explicit than Stevenson's method. The details of the analogous, though simpler, scalar scattering problems are also presented.

I
INTRODUCTION

The purpose of this report is to describe a method of reducing scattering problems to a series of potential problems. We deal with a general class of three dimensional scatterers, smooth, closed, bounded, in short those surfaces for which Green's theorem in any of its guises may be invoked. The solution of a scattering problem, for arbitrary excitation, is expressed as a series in ascending powers of wave number, k. This series is known by a variety of names, including Rayleigh series, quasi-static series, and low frequency expansion. That the first term in such a series could be found as the solution of a potential problem was observed by Rayleigh (1897) who determined this term explicitly for a variety of scatterers of both acoustic and electromagnetic waves. For scalar scattering, the determination of succeeding terms in this series as solutions of potential problems has been described, in varying detail, by Noble (1962), Morse and Feshbach (1953), and Darling and Senior (1965). (See Kleinman (1965a) for a more complete bibliography.)

The derivation of successive terms in this series for electromagnetic scattering was described by Stevenson (1953a). Actually Stevenson described two methods, one for finding the general term in the series and a second special technique for finding the first three terms. All of his specific calculations (Stevenson, 1953b) were carried out using this special technique. No attempt to utilize the general method for obtaining higher order terms has, to this writer's knowledge, been reported, which indicates that if attempts were made, they were unsuccessful. More likely, there were none. This is due to the fact that the analysis is sufficiently involved to discourage most efforts to derive more than three terms in a low frequency expansion (that Stevenson treats the more general case of penetrable scatterers certainly doesn't help). For these, Stevenson's special simpler technique suffices. An attempt to clarify the Stevenson method was made by Senior and Sleator (1964) and the present report may be considered an outgrowth of their work.

The present work demonstrates that the method proposed by Stevenson for finding the general term in the series needs clarification at best and at worst leads to incorrect results. An alternate method, preserving the spirit of Stevenson's approach and indeed largely based on it, is presented which hopefully embodies both clarity and correctness. Conciseness has been sacrificed in an attempt to minimize the chances of further obscuring the subject.

The procedure in the electromagnetic (vector) case is a natural extension of the technique employed in the scalar case. For this reason, and also to introduce some notation as well as concepts in the simplest setting, the next section is devoted to a discussion of how scalar scattering problems may be reduced to the study of a succession of potential problems. In Section 3 we describe Stevenson's method for treating the analogous vector problem and show why it is unsatisfactory. Section 4 presents an alternative to Stevenson's method which eliminates its shortcomings. Section 5 is devoted to an illustrative example.

II
SCALAR SCATTERING

In this section we show how a scalar scattering problem with Dirichlet or Neumann boundary conditions may be reduced to a succession of "standard" potential problems. These terms will be precisely defined as they are introduced.

Let B denote the boundary of a smooth, closed, bounded surface in Euclidian 3-space (or the union of a finite number of such surfaces provided they are disjoint), let $\hat{n}$ denote the outward drawn unit normal at any point of B and let V be the volume exterior to B. Erect a cartesian coordinate system with origin in B and let $\vec{r}$ denote a radius vector to a general point (x, y, z) and $\vec{r}_B$ denote a point on B. Furthermore denote by R the distance between $\vec{r}$ and $\vec{r}_B$, i.e.,

$$R = |\vec{r} - \vec{r}_B| = \sqrt{(x - x_B)^2 + (y - y_B)^2 + (z - z_B)^2} . \qquad (2.1)$$
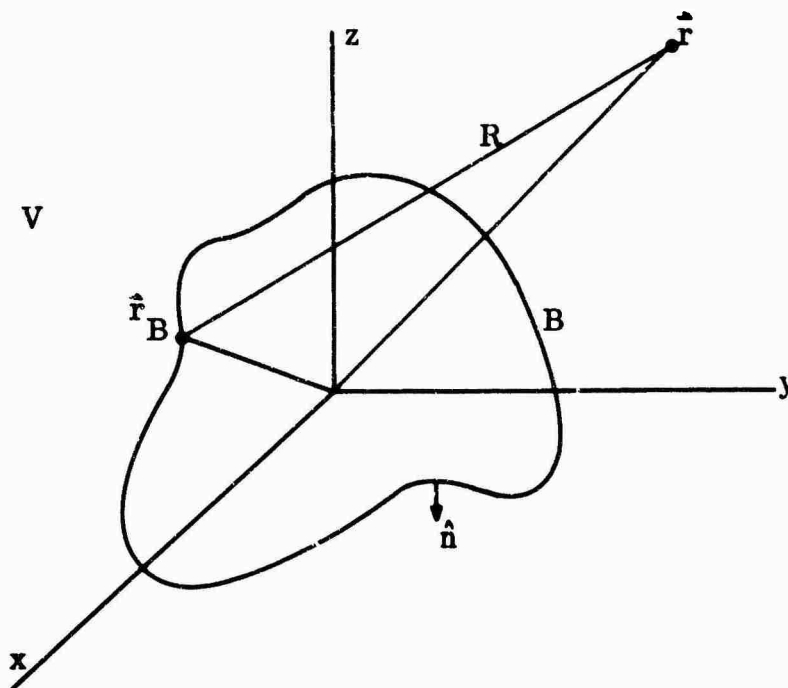
The geometry is illustrated in Fig. 1.

Figure 1

By a scalar scattering problem for the surface B is meant the determination of how the presence of the surface perturbs an incident field, ϕ^{inc}, that is, finding a function $\phi(\vec{r})$ such that

$$(\nabla^2 + k^2)\phi = 0 \qquad\qquad \vec{r} \in V , \qquad\qquad (2.2)$$

$$\lim_{r \to \infty} r\left(\frac{\partial}{\partial r}\phi - ik\phi\right) = 0 , \qquad\qquad (2.3)$$

and either

$$\phi(\vec{r}_B) = -\phi^{inc}(\vec{r}_B) , \qquad\qquad (2.4a)$$

or

$$\left.\frac{\partial\phi(\vec{r})}{\partial n}\right|_{\vec{r}=\vec{r}_B} = -\left.\frac{\partial\phi^{inc}(\vec{r})}{\partial n}\right|_{\vec{r}=\vec{r}_B} \qquad\qquad (2.4b)$$

Equation (2.3) is a statement of Sommerfeld's radiation condition which implies that outgoing waves look like $\dfrac{e^{ikr}}{r} f(\theta, \phi)$ for large r. The boundary conditions (2.4a) and (2.4b) are Dirichlet and Neumann conditions respectively. Specifying either one is sufficient to guarantee the existence of a unique function ϕ hence both the values of the function and its normal derivative may not be assigned arbitrarily. We will consider the Dirichlet and Neumann problems separately but the analysis is quite similar.

The starting point is the Helmholtz integral representation of regular solutions of (2.2); viz,

$$\phi(\vec{r}) = \frac{1}{4\pi} \int_B \left\{ \phi \frac{\partial}{\partial n} \frac{e^{ikR}}{R} - \frac{e^{ikR}}{R} \frac{\partial\phi}{\partial n} \right\} dB . \qquad\qquad (2.5)$$

The integration is carried out over the surface and the normal derivative is $\hat{n} \cdot \nabla$. Next we assume that the unknown function ϕ may be expressed as a conver-

gent power series in k. Actually this need not be assumed, that is, it may be proven that there does exist such an expansion, convergent for k sufficiently small (see Werner, 1962 and Kleinman, 1965b). It should be noted that we are considering k real and positive though the results may be extended to include complex values of k. We write the expansion

$$\phi(\vec{r}) = \sum_{m=0}^{\infty} \phi_m(\vec{r})(ik)^m \tag{2.6}$$

where the factor i is included in the expansion parameter merely as a convenience. The functions ϕ_m are independent of k and each of them may be determined as follows.

Since e^{ikR} is an entire function, the series

$$e^{ikR} = \sum_{l=0}^{\infty} \frac{(ikR)^l}{l!} \tag{2.7}$$

converges for all k. Substituting (2.6) and (2.7) in (2.5), we obtain

$$\sum_{m=0}^{\infty} \phi_m(\vec{r})(ik)^m = \frac{1}{4\pi} \int_B dB \left\{ \sum_{m=0}^{\infty} \phi_m(\vec{r}_B)(ik)^m \frac{\partial}{\partial n} \sum_{l=0}^{\infty} \frac{(ik)^l R^{l-1}}{l!} \right.$$
$$\left. - \sum_{l=0}^{\infty} \frac{(ik)^l R^{l-1}}{l!} \sum_{m=0}^{\infty} \frac{\partial}{\partial n} \phi_m (ik)^m \right\}. \tag{2.8}$$

As long as $|k|$ is strictly less than the radius of convergence we may interchange summation and integration and reorder terms in the double series obtaining

$$\sum_{m=0}^{\infty} \phi_m(\vec{r})(ik)^m = \frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{(ik)^l}{(l-m)!} \int_B \left\{ \phi_m \frac{\partial}{\partial n} R^{l-m-1} - R^{l-m-1} \frac{\partial}{\partial n} \phi_m \right\} dB.$$

$$\tag{2.9}$$

Equating coefficients of like powers of k, with an obvious change in notation, yields

$$\hat{\phi}_\ell(\vec{r}) = \frac{1}{4\pi} \sum_{m=0}^{\ell} \frac{1}{(\ell-m)!} \int_B \left\{ \hat{\phi}_m \frac{\partial}{\partial n} R^{\ell-m-1} - R^{\ell-m-1} \frac{\partial}{\partial n} \hat{\phi}_m \right\} dB , \quad (2.10)$$

$$\ell = 0, 1, 2, \ldots$$

In order to determine $\hat{\phi}_\ell$ we must employ the boundary conditions, hence we must distinguish between the two problems under consideration. Whether the incident field is a plane wave, point source, or linear combination of such sources it remains true that the representation of the incident field is analytic in k. Thus we may write

$$\phi^{inc}(\vec{r}) = \sum_{\ell=0}^{\infty} \phi_\ell^{inc}(\vec{r})(ik)^\ell . \qquad (2.11)$$

The boundary conditions (2.4a) and (2.4b) then imply that either

$$\hat{\phi}_\ell(\vec{r}_B) = -\phi_\ell^{inc}(\vec{r}_B) \qquad (2.12a)$$

or

$$\left. \frac{\partial \hat{\phi}_\ell}{\partial n} \right|_{\vec{r}=\vec{r}_B} = - \frac{\partial}{\partial n} \phi_\ell^{inc}(\vec{r}) \bigg|_{\vec{r}=\vec{r}_B} \qquad (2.12b)$$

Consider first the Dirichlet problem, (2.12a). Inserting the boundary values in the integral representation (2.10) produces the system of equations

$$\hat{\phi}_\ell(\vec{r}) = - \frac{1}{4\pi} \sum_{m=0}^{\ell} \frac{1}{(\ell-m)!} \int_B \left\{ \phi_m^{inc} \frac{\partial}{\partial n} R^{\ell-m-1} + R^{\ell-m-1} \frac{\partial}{\partial n} \hat{\phi}_m \right\} dB$$

$$(2.13)$$

We treat first the case when $\ell = 0$. Equation (2.13) becomes simply

$$\Phi_0(\vec{r}) = -\frac{1}{4\pi}\int_B \Phi_0^{inc} \frac{\partial}{\partial n} \frac{1}{R} dB - \frac{1}{4\pi}\int_B \frac{1}{R} \frac{\partial \Phi_0}{\partial n} dB .\qquad (2.14)$$

The unknown term on the right is clearly an exterior potential function or in the language of potential theory (e.g. Kellog, 1929) the field of a single layer distribution of density $\partial \Phi_0/\partial n$. That is, if we designate by ϕ_0 the unknown function,

$$\phi_0(\vec{r}) = -\frac{1}{4\pi}\int_B \frac{1}{R} \frac{\partial \Phi_0}{\partial n} dB ,\qquad (2.15)$$

then ϕ_0 satisfies the equation

$$\nabla^2 \phi_0(\vec{r}) = 0 , \qquad \vec{r} \epsilon V ,\qquad (2.16)$$

and ϕ_0 is regular at infinity in the sense of Kellog, viz.

$$\lim_{r \to \infty} \left| r\phi_0 \right| < \infty , \qquad \lim_{r \to \infty} \left| r^2 \frac{\partial \phi_0}{\partial r} \right| < \infty .\qquad (2.17)$$

Furthermore, with the boundary condition (2.12a) and the expression (2.14), the values of ϕ_0 on B are specified, i.e.,

$$\phi_0(\vec{r}_B) = -\Phi_0^{inc}(\vec{r}_B) + \lim_{r \to r_B} \frac{1}{4\pi}\int_B \Phi_0^{inc} \frac{\partial}{\partial n}\left(\frac{1}{R}\right) dB \qquad (2.18)$$

Note that the integration in (2.18) must be carried out before the limit is taken so the integrand is always defined. With this proviso the right hand side of (2.18) is well behaved and completely specified in terms of the incident field. Equations (2.16)-(2.18) constitute a standard exterior Dirichlet potential problem which has a unique solution. Next we show that succeeding terms Φ_ℓ may be written in terms of solutions of similar problems. To this end assume that $\Phi_0, \Phi_1, \ldots \Phi_{\ell-1}$ are all

known. Then (2.13) may be written

$$\bar{\phi}_\ell(\vec{r}) = F_\ell(\vec{r}) + \phi_\ell(\vec{r}) \tag{2.19}$$

where

$$F_\ell(\vec{r}) = -\frac{1}{4\pi} \sum_{m=0}^{\ell} \frac{1}{(\ell-m)!} \int_B \bar{\phi}_m^{inc} \frac{\partial}{\partial n} R^{\ell-m-1} dB$$

$$-\frac{i}{4\pi} \sum_{m=0}^{\ell-1} \frac{1}{(\ell-m)!} \int_B R^{\ell-m-1} \frac{\partial \bar{\phi}_m}{\partial n} dB$$

and

$$\phi_\ell(\vec{r}) = -\frac{1}{4\pi} \int_B \frac{1}{R} \frac{\partial \bar{\phi}_\ell}{\partial n} dB .$$

With the assumption that all $\bar{\phi}$'s are know up to, but not including $\bar{\phi}_\ell$, $F_\ell(\vec{r})$ is a known function. Clearly $\phi_\ell(\vec{r})$ is again a single layer distribution, satisfies (2.16) and (2.17), and is uniquely determined with the boundary condition

$$\phi_\ell(\vec{r}_B) = -\bar{\phi}_\ell^{inc}(\vec{r}_B) - F_\ell(\vec{r}_B) \tag{2.20}$$

Again, care must be used in letting $\vec{r} \to \vec{r}_B$ in one term of F_ℓ but there is no intrinsic difficulty. Thus $\bar{\phi}_\ell$ is determined in terms of a known function F_ℓ and a solution of a standard exterior Dirichlet potential problem, ϕ_ℓ. We have shown that is true for $\ell = 0$, and also for $\ell > 0$ provided $\bar{\phi}_{\ell-1}$, $\bar{\phi}_{\ell-2}$, ... $\bar{\phi}_o$ have previously been found. The solution of the Dirichlet scattering problem is then given by (2.6).

An exactly analogous procedure may be followed for the Neumann boundary condition, (2.12b). Corresponding to (2.13) we have

$$\Phi_\ell(\vec{r}) = \frac{1}{4\pi} \sum_{m=0}^{\ell} \frac{1}{(\ell-m)!} \int_B \left\{ R^{\ell-m-1} \frac{\partial}{\partial n} \Phi_m^{inc} + \Phi_m \frac{\partial}{\partial n} R^{\ell-m-1} \right\} dB \qquad (2.21)$$

Furthermore this may be written

$$\Phi_\ell(\vec{r}) = G_\ell(\vec{r}) + \psi_\ell(\vec{r}) \qquad (2.22)$$

where

$$G_\ell(\vec{r}) = \frac{1}{4\pi} \sum_{m=0}^{\ell} \frac{1}{(\ell-m)!} \int_B R^{\ell-m-1} \frac{\partial}{\partial n} \Phi_m^{inc} \, dB$$

$$+ \frac{1}{4\pi} \sum_{m=0}^{\ell-1} \frac{1}{(\ell-m)!} \int_B \Phi_m \frac{\partial}{\partial n} R^{\ell-m-1} \, dB \,,$$

the second sum is identically zero if $\ell = 0$, and

$$\psi_\ell(\vec{r}) = \frac{1}{4\pi} \int_B \Phi_\ell \frac{\partial}{\partial n} \left(\frac{1}{R} \right) dB \,.$$

G_ℓ is completely determined if $\Phi_{\ell-1}, \ldots \Phi_0$ are known and $\psi_\ell(\vec{r})$ is a double layer distribution. That is, ψ_ℓ is the solution of a standard exterior Neumann potential problem, namely,

$$\nabla^2 \psi_\ell(\vec{r}) = 0 \,, \qquad \vec{r} \epsilon V$$

ψ_ℓ regular in the sense of Kellog, (2.17),

and

$$\left. \frac{\partial \psi_\ell}{\partial n} \right|_{\vec{r}=\vec{r}_B} = - \left. \frac{\partial \Phi_\ell^{inc}}{\partial n} \right|_{r=r_B} - \left. \frac{\partial G_\ell}{\partial n} \right|_{\vec{r}=\vec{r}_B}$$

We have thus demonstrated that for either Dirichlet or Neumann scattering problems successive terms in the low frequency expansion may be determined by solving a succession of standard potential problems. That is, the first term is the solution of such a potential problem the second term is expressed in terms of the first and the solution of a potential problem, the third is given in terms of the first two and a potential solution, etc.

Before closing this section, a word should be said about low frequency expansions of the far field. The Rayleigh series, (2.6), may be considered as an expansion of the near field which, if all terms are included, is also valid in the far field. If only a finite number of terms are known, then the truncated series does not in itself give much useful information about the far field. Such information is available if we again make use of the integral representation (2.5). To this end note that for large r

$$\frac{e^{ikR}}{R} \sim \frac{e^{ikr - ik\frac{\vec{r} \cdot \vec{r}_B}{r}}}{r} = \frac{e^{ikr - ik\hat{r} \cdot \vec{r}_B}}{r} \quad , \quad r = |\vec{r}|, \quad \hat{r} = \vec{r}/r \quad (2.23a)$$

and

$$\nabla \frac{e^{ikR}}{R} \sim ik\hat{r} \frac{e^{ikr - ik\hat{r} \cdot \vec{r}_B}}{r} \quad (2.23b)$$

If (r, θ, ϕ) and (r_B, θ_B, ϕ_B) are spherical coordinates of points $\vec{r}$ and $\vec{r}_B$ respectively then

$$\hat{r} \cdot \vec{r}_B = r_B \left[\cos\theta \cos\theta_B + \sin\theta \sin\theta_B \cos(\phi - \phi_B) \right] . \quad (2.24)$$

Substituting (2.23a) and (2.23b) in (2.5) we obtain, for large r,

$$\phi(\vec{r}) \sim \frac{e^{ikr}}{4\pi r} \int_B \left\{ \phi ik\hat{n} \cdot \hat{r} e^{-ik\hat{r} \cdot \vec{r}_B} - e^{-ik\hat{r} \cdot \vec{r}_B} \frac{\partial \phi}{\partial n} \right\} dB . \quad (2.25)$$

Now if we substitute expansions of ϕ and $e^{-ik\hat{r}\cdot\vec{r}_B}$ in the right hand side of (2.25) and rearrange terms we obtain

$$\phi(\vec{r}) \sim \frac{e^{ikr}}{4\pi r} \sum_{\ell=0}^{\infty} (ik)^{\ell} \sum_{m=0}^{\ell} \frac{(-1)^{\ell-m}}{(\ell-m)!} \int_B (\hat{r}\cdot\vec{r}_B)^{\ell-m} \left(\hat{n}\cdot\hat{r}\,\phi_{m-1} - \frac{\partial\phi_m}{\partial n} \right) dB$$

(2.26)

where $\phi_{-1} \equiv 0$.

Examination of equation (2.26) reveals that knowledge of a finite number of terms in the low frequency expansion of the near field (the ϕ_m's) provides similar information about the low frequency expansion of the far field coefficient, i.e. the coefficient of e^{ikr}/r. More specifically, in the Dirichlet case, when the boundary conditions specify ϕ_m on B for all m, then knowledge of the first ℓ $\frac{\partial\phi_m}{\partial n}$'s $\left(\frac{\partial\phi_o}{\partial n},\ \frac{\partial\phi_1}{\partial n},\ \ldots \frac{\partial\phi_{\ell-1}}{\partial n} \right)$ will provide, with equation (2.26), the first ℓ terms of the far field expansion. In the Neumann case, ($\frac{\partial\phi_m}{\partial n}$ on B given for all m) the first ℓ ϕ_m's will apparently give $\ell+1$ terms in the far field. However, it may be shown that, whether ϕ^{inc} is a plane wave or a point source,

$$\int_B \frac{\partial\phi_o^{inc}}{\partial n}\ dB = 0$$

(2.27)

hence only ℓ terms in the far field are specified. Or another way of saying this is that, in the Neumann problem, ℓ near field terms produce $\ell+1$ far field terms but the first term, i.e., the coefficient of $(ik)^0$, is always zero.

III
STEVENSON'S METHOD

In this section we shall describe Stevenson's attempt to generalize the approach of Section 2 to electromagnetic scattering and pay particular attention to the shortcomings, rather than the strong points (which are numerous) of Stevenson's work. To effect some simplification, we shall treat only the case of scattering by a perfectly conducting surface whereas Stevenson considered more general scatterers. It seems clear, however, that both the criticism in this section and the correction in the following section may be applied in the more general case.

The surface geometry and notation are the same as introduced in Section 2 and depicted in Fig. 1, which is here reproduced for convenience.
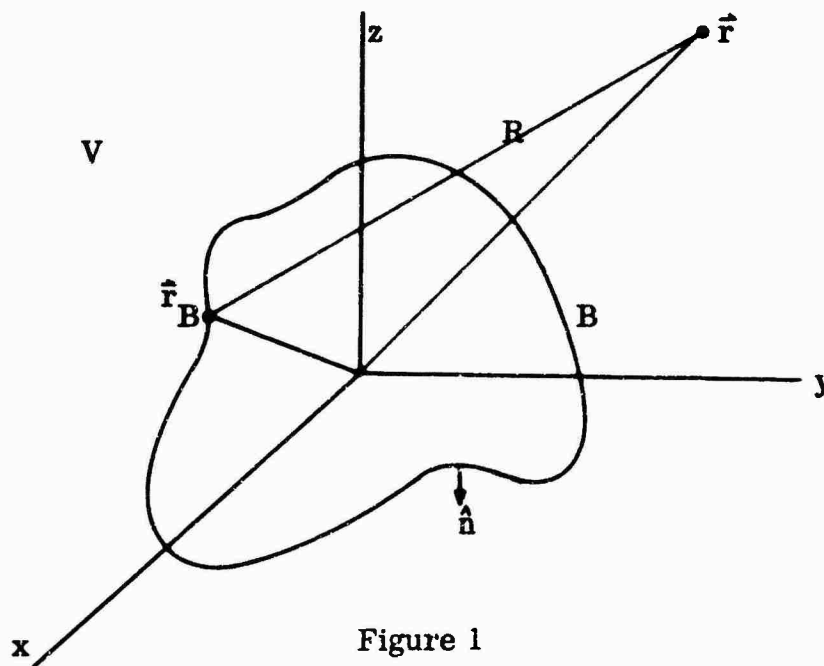
Figure 1

By an electromagnetic scattering problem for the perfectly conducting surface B is meant the problem of determining how the presence of the surface perturbs an incident electromagnetic field, $(\vec{E}^{inc}, \vec{H}^{inc})$. That is, we seek a solution of Maxwell's equations

$$\nabla \times \vec{E} = ik\vec{H}, \qquad \nabla \cdot \vec{E} = 0, \qquad \hat{\vec{r}} \in V \qquad (3.1)$$

$$\nabla \times \vec{H} = -ik\vec{E}, \qquad \nabla \cdot \vec{H} = 0$$

subject to the boundary conditions

$$\hat{n} \times \vec{E}\Big|_{r=\vec{r}_B} = -\hat{n} \times \hat{E}^{inc}\Big|_{\vec{r}=\vec{r}_B} , \qquad \hat{n} \cdot \vec{H}\Big|_{r=r_B} = -\hat{n} \cdot \vec{H}^{inc}\Big|_{\vec{r}=\hat{r}_B} \qquad (3.2)$$

and the radiation condition

$$\lim_{r \to \infty} \vec{r} \times (\nabla \times \vec{E}) \div ikr\vec{E} = \lim_{r \to \infty} \vec{r} \times (\nabla \times \vec{H}) + ikr\vec{H} = 0 \qquad (3.3)$$

<div align="right">uniformly in $\hat{r}$</div>

(The divergence conditions and boundary condition on $\vec{H}$ are redundant, i.e. may be deduced from the other conditions.)

In attempting to show how to reduce this problem to that of solving a series of potential problems, the procedure parallels that followed in the scalar problem. Corresponding to the Helmholtz integral representation (2.2) we employ the expression derived by Stratton and Chu (see Stratton, 1941) which expresses the field at any exterior point in terms of its values on the surface B. [Wilcox (1956) also derives these formulas but strangely omits any reference to the Stratton-Chu work.]

$$\vec{E}(\vec{r}) = \frac{1}{4\pi} \nabla \times \int_B \frac{e^{ikR}}{R} \hat{n} \times \vec{E} \, dB + \frac{ik}{4\pi} \int_B \frac{e^{ikR}}{R} \hat{n} \times \vec{H} \, dB - \frac{1}{4\pi} \nabla \int_B \frac{e^{ikR}}{R} \hat{n} \cdot \vec{E} \, dB$$

<div align="right">(3.4a)</div>

$$\vec{H}(\vec{r}) = \frac{1}{4\pi} \nabla \times \int_B \frac{e^{ikR}}{R} \hat{n} \times \vec{H} \, dB - \frac{ik}{4\pi} \int_B \frac{e^{ikR}}{R} \hat{n} \times \vec{E} \, dB - \frac{1}{4\pi} \nabla \int_B \frac{e^{ikR}}{R} \hat{n} \cdot \vec{H} \, dB .$$

<div align="right">(3.4b)</div>

Recall that R is a function of the coordinates of two points $\vec{r}$ and $\vec{r}_B$, everything else in the integrands on the right hand sides is a function of the integration variables (coordinates of $\vec{r}_B$) and ∇ operates on $\vec{r}$. For future use, we denote by ∇_B the operator on $\vec{r}_B$ and note that

$$\nabla \frac{e^{ikR}}{R} = -\nabla_B \frac{e^{ikR}}{R} \quad . \tag{3.5}$$

Now following Stevenson as well as the procedure in the scalar case we assume that $\vec{E}$ and $\vec{H}$ may be expanded in series of powers of k, i.e.,

$$\vec{E}(\vec{r}) = \sum_{m=0}^{\infty} \vec{E}_m(\vec{r})(ik)^m, \qquad \vec{H}(\vec{r}) = \sum_{m=0}^{\infty} \vec{H}_m(\vec{r})(ik)^m \quad . \tag{3.6}$$

As before, this assumption has been proven (Werner, 1963), that is, it is no longer an assumption but a consequence of (3.1), (3.2) and (3.3). It is perhaps worthy of note that the reason this entire discussion concerns three-dimensional scattering problems is that convergent expansions of the form (3.6) do not exist for two-dimensional scattered fields.

Next we expand the free space Green's function, e^{ikR}/R, in a series, viz,

$$\frac{e^{ikR}}{R} = \sum_{\ell=0}^{\infty} \frac{(ik)^\ell R^{\ell-1}}{\ell!} \quad , \tag{3.7}$$

then substitute (3.6) and (3.7) in (3.4a), (3.4b). After interchanging summation and integration, reordering terms and equating like powers of (ik), we obtain

$$\vec{E}_\ell(\vec{r}) = \frac{1}{4\pi} \nabla \times \sum_{m=0}^{\ell} \frac{1}{m!} \int_B \hat{n} \times \vec{E}_{\ell-m} R^{m-1} dB$$

$$+ \frac{1}{4\pi} \sum_{m=0}^{\ell-1} \frac{1}{m!} \int_B \hat{n} \times \vec{H}_{\ell-m-1} R^{m-1} dB - \frac{1}{4\pi} \sum_{m=0}^{\ell} \frac{1}{m!} \nabla \int_B \hat{n} \cdot \vec{E}_{\ell-m} R^{m-1} dB$$

$$\ell = 0, 1, 2, \ldots \tag{3.8a}$$

$$\vec{H}_\ell(\vec{r}) = \frac{1}{4\pi} \nabla \times \sum_{m=0}^{\ell} \frac{1}{m!} \int_B \hat{n} \times \vec{H}_{\ell-m} R^{m-1} dB - \frac{1}{4\pi} \sum_{m=0}^{\ell-1} \frac{1}{m!} \int_B \hat{n} \times \vec{E}_{\ell-m-1} R^{m-1} dB$$

$$- \frac{1}{4\pi} \sum_{m=0}^{\ell} \frac{1}{m!} \nabla \int_B \hat{n} \cdot \vec{H}_{\ell-m} R^{m-1} dB \qquad (3.8b)$$

where $\displaystyle\sum_{m=0}^{\ell-1} \equiv 0$ when $\ell = 0$.

Furthermore, substituting the series (3.6) in Maxwell's equations (3.1) yields

$$\nabla \times \vec{E}_o = 0 \qquad\qquad \nabla \times \vec{H}_o = 0 \qquad\qquad\qquad (3.9)$$

$$\nabla \times \vec{E}_\ell = \vec{H}_{\ell-1} \qquad\qquad \nabla \times \vec{H}_\ell = -\vec{E}_{\ell-1} \qquad \ell = 1, 2, 3 \ldots \qquad (3.10)$$

$$\nabla \cdot \vec{E}_\ell = 0 \qquad\qquad \nabla \cdot \vec{H}_\ell = 0 \qquad\qquad \ell = 0, 1, 2, 3 \qquad (3.11)$$

and the boundary conditions (3.2) become

$$\hat{n} \times \vec{E}_\ell \Big|_{\vec{r}=\vec{r}_B} = -\hat{n} \times \vec{E}_\ell^{inc}\Big|_{\vec{r}=\vec{r}_B} \quad , \quad \hat{n} \cdot \vec{H}_\ell\Big|_{\vec{r}=\vec{r}_B} = -\hat{n} \cdot \vec{H}_\ell^{inc}\Big|_{\vec{r}=\vec{r}_B} \quad (3.12)$$

These last equations result from the fact that, as with scalar sources, representations of electromagnetic plane waves or point sources are analytic in k. There is one more condition of importance. With Maxwell's equations and Stokes' theorem it is a simple matter to show that

$$\int_B \hat{n} \cdot \vec{E} \, dB = 0 \; , \qquad\qquad \int_B \hat{n} \cdot \vec{H} \, dB = 0 \qquad\qquad (3.13)$$

It follows then from the series expansions (3.6) that

$$\int_B \hat{n} \cdot \vec{E}_\ell \, dB = 0 \; , \qquad\qquad \int_B \hat{n} \cdot \vec{H}_\ell \, dB = 0 \qquad\qquad (3.14)$$
$$\ell = 0, 1, 2, 3 \ldots$$

Stevenson then proceeds to show how the zeroth order terms, $\vec{E}_o$ and $\vec{H}_o$, may be determined as solutions of potential problems. For the perfectly conducting case this reduction to potential problems for the zeroth order terms will be included in the general treatment of the following section, and, since we have no quarrel with Stevenson's results for these terms, the details will be omitted here. To calculate higher order terms, Stevenson proposes the following procedure: Suppose $\vec{E}_o, \vec{E}_1, \ldots \vec{E}_{\ell-1}, \vec{H}_o, \vec{H}_1, \ldots \vec{H}_{\ell-1}$ are known. To find $\vec{E}_\ell$ or $\vec{H}_\ell$, determine first a particular solution of (3.10), that is, find functions $\vec{F}_\ell$ and $\vec{G}_\ell$ such that

$$\nabla \times \vec{F}_\ell = \vec{H}_{\ell-1} \qquad \text{and} \qquad \nabla \times \vec{G}_\ell = -\vec{E}_{\ell-1}$$

The differences between these particular solutions and the true coefficients, $\vec{E}_\ell - \vec{F}_\ell$ and $\vec{H}_\ell - \vec{G}_\ell$, are gradients of unknown potential functions (not necessarily regular at infinity) i.e.,

$$\vec{E}_\ell = \vec{F}_\ell + \nabla \phi_\ell \tag{3.16a}$$

$$\vec{H}_\ell = \vec{G}_\ell + \nabla \psi_\ell \tag{3.16b}$$

Substitute (3.16a) and (3.16b) into the integral expressions (3.8a) and (3.8b) respectively, also introduce the boundary conditions, (3.12). There results equations for $\vec{E}_\ell$ and $\vec{H}_\ell$ which contain some known terms and some unknown. It is then possible to show that the unknown terms are now <u>exterior</u> potential functions (regular at infinity) which may be determined as solutions of standard potential problems.

The process, once begun, appears to be both correct and, in the details of its execution, ingenious. The source of trouble, however, is right at the beginning; namely how does one determine particular solutions of the equations

$$\nabla \times \vec{E}_\ell = \vec{H}_{\ell-1} \, , \qquad \nabla \times \vec{H}_\ell = -\vec{E}_{\ell-1} \, ?$$

In a separate paper, Stevenson (1954) points out that necessary and sufficient conditions for the equation

$$\nabla \times \vec{F}(\vec{r}) = \vec{f}(\vec{r}) \ , \qquad \vec{r} \in V \tag{3.17}$$

to have a solution are

$$\nabla \cdot \vec{f} = 0 \tag{3.18}$$

and

$$\int_B \hat{n} \cdot \vec{f} \, ds = 0 \tag{3.19}$$

where, if B consists of a number of disjoint surfaces, $B_1, \ldots B_i$, then (3.19) must hold for each separately, as well as the sum. With this we have no quarrel. Stevenson then goes on to assert that an explicit solution of the problem is given by

$$F(\vec{r}) = \frac{1}{4\pi} \ \nabla \times \int_{\substack{all \\ space}} \frac{f(\vec{r}_v)}{R(\vec{r}, \vec{r}_v)} \ dv \quad , \tag{3.20}$$

provided that $\vec{f}$ satisfies (3.18) and (3.19).

Since the integration is over all space, not merely V, the exterior of B, this expression requires some explanation. In the first place, $f(\vec{r})$ is originally defined only exterior to B. To extend the definition to the interior, Stevenson proposes to choose $\vec{f}$ so that (3.18) remains true and that $\hat{n} \cdot \vec{f}$ is continuous at B. This he accomplishes by choosing

$$\vec{f}(\vec{r}) = \nabla u(\vec{r}) \ , \qquad \vec{r} \text{ interior to } B \tag{3.21}$$

where

$$\nabla^2 u = 0 \ , \qquad \vec{r} \text{ in } B \tag{3.22}$$

and

$$\hat{n} \cdot \nabla u \Big|_{\vec{r}=\vec{r}_B} \equiv \frac{\partial u}{\partial n} \Big|_{\vec{r}=\vec{r}_B} = \hat{n} \cdot \vec{f} \Big|_{\vec{r}=\vec{r}_B} \qquad (3.23)$$

This is a standard interior Neumann potential problem for u and has a unique solution provided that

$$\int_B \frac{\partial u}{\partial n} \, dB = 0 \ . \qquad (3.24)$$

That (3.24) holds is guaranteed by (3.19). Thus the extension to the interior is carried out, once this potential problem is solved. Equation (3.20) then is the required solution provided the integral exists, that is, provided

$$\vec{f} = O(1/r^3) \qquad \text{as } r \rightarrow \infty \ . \qquad (3.25)$$

Stevenson describes the proof and we shall demonstrate it in detail in the following section where we again make use of this device. Now however, we accept it and finally get to the heart of the matter, namely, what do we do if f is defined originally in the infinite region V, but does not satisfy the necessary order condition at infinity, equation (3.25)? This in fact is exactly what happens since $\vec{E}_\ell$ and $\vec{H}_\ell$ vanish as $1/r^3$ only for $\ell = 0$ which allows us, using the method described, to determine $\vec{E}_1$ and $\vec{H}_1$ but apparently no higher order terms. (Actually we may go one term further since the $1/r^2$ terms don't contribute to the integral.) Stevenson was aware of this and proposed the following procedure:

If V is the unbounded region exterior to B and if $\vec{f}$ does not vanish at infinity to the required order, first surround B by a surface B_o. Then redefine $\vec{f}$ exterior to B_o in terms of the solution of an exterior potential problem, namely, let

$$\vec{f}(\vec{r}) = \nabla u, \qquad \vec{r} \text{ exterior to } B_o \qquad (3.26)$$

where
$$\nabla^2 u = 0$$

$$\left.\frac{\partial u}{\partial n}\right|_{\vec{r}=\vec{r}_{B_o}} = \left.\hat{n} \cdot \vec{f}\right|_{\vec{r}=\vec{r}_{B_o}} \qquad (3.27)$$

u regular at infinity.

This problem has a unique solution u and, since

$$\int_{B_o} \hat{n} \cdot \vec{f} dB_o = \int_{B_o} \frac{\partial u}{\partial n} dB_o = 0 \qquad ^* \qquad (3.28)$$

it follows that $u = O(1/r^2)$ hence $\vec{f}$ will satisfy (3.25). With $\vec{f}$ thus redefined, the solution (3.20) exists and is valid in the portion of V interior to B_o where B_o can be taken arbitrarily large.

With that, Stevenson apparently considers the subject closed. The implication is that since B_o may be taken arbitrarily large we may take it as a sphere whose radius becomes infinite and then (3.20) will represent the solution we seek throughout V. But, unfortunately, if f were a function whose original behavior at infinity was insufficient to guarantee existence of the integral in (3.20), then the limit of the integral with $\vec{f}$ redefined may not exist as the radius of B_o becomes infinite. This argument by which the unpleasant behavior at infinity is avoided (that is, confining attention to a finite volume, carrying out the calculation, and then letting the volume become infinite) is not only employed by Stevenson but others as well, e.g. Morse and Feshbach (1953, I, p. 53). It does produce the desired results in many cases. For example, the process is valid whenever f is the gradient of a potential function, regardless of its behavior at infinity (which includes the example

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*If B_o is any surface entirely containing B and equations (3.18) and (3.19) hold, then (3.28) follows from Gauss' theorem relating volume and surface integrals.

used by Morse and Feshbach). That it may also yield unacceptable results is illustrated in the following example, where f is not the gradient of a potential function though still satisfies (3.18) and (3.19). This indeed is representative of the behavior one would encounter in actually attempting to find particular solutions of (3.10).

Let

$$\vec{f}(\vec{r}) = \nabla x r^m \hat{i}_x = mr^{m-2}(z\hat{i}_y - y\hat{i}_z) , \tag{3.29}$$

$$r = \sqrt{x^2 + y^2 + z^2} .$$

Clearly a particular solution of $\nabla x \vec{F} = \vec{f}$ is

$$\vec{F} = r^m \hat{i}_x \tag{3.30}$$

However, let us attempt to determine a particular solution using equation (3.20). First of all it is a trivial calculation to observe that (3.18) and (3.19) are satisfied with this particular $\vec{f}$. For this simple example we have no scattering surface B, but with Gauss' theorem it is clear that for any closed surface B, $\int_B \hat{n} \cdot \vec{f} dB = 0$. Furthermore, the function $\vec{f}$ clearly misbehaves at infinity so that to use (3.20) we must employ the redefinition of $\vec{f}$. Thus choose B_o to be a large sphere of radius r_o. Next define

$$\begin{aligned} \underset{\sim}{\vec{f}} &= \vec{f} , & r \leqslant r_o \\ &= \nabla u , & r \geqslant r_o \end{aligned} \tag{3.31}$$

where

$$\nabla^2 u = 0 \qquad r \geqslant r_o$$

$$\hat{n} \cdot \nabla u \Big|_{r=r_o} = \hat{n} \cdot \vec{f} \Big|_{r=r_o}$$

u regular at infinity.

Then

$$F(\vec{r}) = \frac{1}{4\pi} \; \nabla x \int_{\substack{all \\ space}} \frac{\vec{\underset{\sim}{f}}(\vec{r}_v)}{|\vec{r} - \vec{r}_o|} \; dv \; . \tag{3.32}$$

Note that with our choice of B_o and $\vec{f}$,

$$\hat{n} \cdot \vec{f} \bigg|_{r=r_o} = \hat{r} \cdot \nabla x r^m \hat{i}_x = \hat{r} \cdot mr_o^{m-1}(\hat{\theta}\sin\phi + \hat{\phi}\cos\theta\cos\phi) = 0. \tag{3.33}$$

Hence u is a solution of the homogeneous Laplace equation, regular at infinity, satisfying homogeneous boundary conditions on B_o which means that

$$u \equiv 0 \tag{3.34}$$

Thus

$$\int_{\substack{all \\ space}} \frac{\vec{\underset{\sim}{f}}(\vec{r}_v)}{|\vec{r} - \vec{r}_v|} \; dv = \int_{r_v \leqslant r_o} \frac{\vec{f}(\vec{r}_v)}{|\vec{r} - \vec{r}_v|} \; dv \tag{3.35}$$

This integration is easily performed yielding

$$\int_{r_v \leqslant r_o} \frac{\vec{f}(r_v)}{|\vec{r} - \vec{r}_v|} \; dv = \int_0^{r_o} dr_v \int_0^{2\pi} d\phi_v \int_0^{\pi} d\theta_v \, r_v^2 \sin\theta_v \frac{\nabla_v x r_v^m}{|\vec{r} - \vec{r}_v|} \hat{i}_x$$

$$= \frac{4\pi}{3}\left(r_o^m - \frac{3r^m}{m+3}\right)(z\hat{i}_y - y\hat{i}_z), \; r \leqslant r_o \tag{3.36}$$

$$= \frac{4\pi}{3} \frac{m}{m+3} \frac{r_o^{m+3}}{r^3} (z\hat{i}_y - y\hat{i}_z), \; r \geqslant r_o \; .$$

Now we form $\vec{F}$ using (3.32) and find

$$\vec{F}(\vec{r}) = -\frac{2}{3} r_o^m \hat{i}_x + \frac{m+2}{m+3} r^m \hat{i}_x - \frac{m \, x \, r^{m-1}}{m+3} \hat{r} \qquad r < r_o$$

(3.37)

$$= \frac{m r_o^{m+3}}{3(m+3)r^3} (\hat{i}_x - 3 \frac{x}{r} \hat{r}) \qquad r > r_o$$

It is a simple calculation to show that this $\vec{F}$ is indeed a solution of $\nabla \times \vec{F} = \hat{f}$ when $r < r_o$. However, it is also clear that $\vec{F}$ as defined in (3.37) does not exist as $r_o \to \infty$. Furthermore, if r_o remains finite, then the function $\vec{F}$ not only exhibits an unwanted dependence on an arbitrary parameter (the radius r_o of B_o) but also is discontinuous on B_o. This violates the tacit requirement that $\vec{F}$ be a differentiable solution of $\nabla \times \vec{F} = \vec{f}$ for all points in V.

How then do we proceed in those cases when Stevenson's scheme for finding particular solutions apparently fails? One method would be to attempt to show that the undesirable part of $\vec{F}$ is the gradient of a scalar function and can therefore be neglected; the remaining part of $\vec{F}$ would still be a solution of $\nabla \times \vec{F} = \vec{f}$. In the example above it is easily seen that $\vec{F}$ may be written

$$\vec{F} = \frac{m+2}{m+3} r^m \hat{i}_x - \frac{m \, x \, r^{m-1}}{m+3} \hat{r} + \nabla \left(-\frac{2}{3} r_o^m x \right)$$

(3.38)

hence a particular solution of $\nabla \times \vec{F} = \vec{f}$, valid everywhere in V may be obtained merely by deleting the term $\nabla \left(-\frac{2}{3} r_o^m x \right)$. In general, however, the process of identifying the unwanted terms with the gradient of some functions may not be so easily accomplished and in any event adds yet another complication to an already involved procedure.

Rather than attempt to prove that this procedure can be made correct in the manner indicated, we shall end this section having demonstrated that, as it stands, Stevenson's procedure is ambiguous. In the next section we shall show that this

problem of finding particular solutions of (3.10) may be avoided entirely and the process of determining successive terms in the low frequency expansion may be made more straightforward.

IV
ELECTROMAGNETIC SCATTERING – AN ALTERNATE APPROACH

In this section we again treat the problem of extending to electromagnetic scattering the method of Section 2 whereby scalar scattering problems are reduced to a series of standard potential problems. Though the method described here departs from Stevenson's approach, the debt to his work, both in ideas and technique, is large.

We formulate the problem exactly as in Section 3 and the details will not be repeated. The starting point for this analysis is the integral representation of the coefficients in the low frequency expansions of the scattered field, equations (3.8a) and (3.8b). That is, we write the field scattered from the perfectly conducting body, B, as

$$\vec{E}(\vec{r}) = \sum_{\ell=0}^{\infty} \vec{E}_\ell(\vec{r})(ik)^\ell \ , \qquad \vec{H}(\vec{r}) = \sum_{\ell=0}^{\infty} \vec{H}_\ell(\vec{r})(ik)^\ell \ , \qquad (4.1)$$

then the boundary conditions at the surface B are

$$\hat{n} \times \vec{E}_\ell = -\hat{n} \times \vec{E}_\ell^{inc} \ , \qquad \hat{n} \cdot \vec{H}_\ell = -\hat{n} \cdot \vec{H}_\ell^{inc} \ , \qquad \ell = 0, 1, 2 \ldots \qquad (4.2)$$

and, furthermore,

$$\int_B \hat{n} \cdot \vec{E}_\ell dB = 0 \qquad \int_B \hat{n} \cdot \vec{H}_\ell dB = 0 \qquad \ell = 0, 1, 2, \ldots . \qquad (4.3)$$

The Stratton-Chu integral representation, after expanding in powers of k, equating coefficients and using the boundary conditions may be written

$$\vec{E}_\ell(\vec{r}) = \vec{F}_\ell(\vec{r}) - \frac{1}{4\pi} \nabla \int_B \frac{\hat{n} \cdot \vec{E}_\ell}{R} dB \qquad (4.4a)$$

$$\vec{H}_\ell(\vec{r}) = \vec{G}_\ell(\vec{r}) + \frac{1}{4\pi} \nabla \times \int_B \frac{\hat{n} \times \vec{H}_\ell}{R} dB \tag{4.4b}$$

where

$$\vec{F}_\ell(\vec{r}) = -\frac{1}{4\pi} \nabla \times \sum_{m=0}^{\ell} \frac{1}{m!} \int_B \hat{n} \times \vec{E}_{\ell-m}^{inc} R^{m-1} dB \tag{4.5a}$$

$$+ \frac{1}{4\pi} \sum_{m=0}^{\ell-1} \frac{1}{m!} \int_B \hat{n} \times \vec{H}_{\ell-m-1} R^{m-1} dB - \frac{1}{4\pi} \sum_{m=1}^{\ell} \frac{1}{m!} \nabla \int_B \hat{n} \cdot \vec{E}_{\ell-m} R^{m-1} dB$$

and

$$\vec{G}_\ell(\vec{r}) = \frac{1}{4\pi} \sum_{m=1}^{\ell} \frac{1}{m!} \nabla \times \int_B \hat{n} \times \vec{H}_{\ell-m} R^{m-1} dB \tag{4.5b}$$

$$+ \frac{1}{4\pi} \sum_{m=0}^{\ell-1} \frac{1}{m!} \int_B \hat{n} \times \vec{E}_{\ell-m-1}^{inc} R^{m-1} dB + \frac{1}{4\pi} \sum_{m=0}^{\ell} \frac{1}{m!} \nabla \int_B \hat{n} \cdot \vec{H}_{\ell-m}^{inc} R^{m-1} dB.$$

Equations (4.4a, b) and (4.5a, b) hold for all $\ell = 0, 1, 2, \ldots$, however, the terms $\sum_{m=0}^{\ell-1}$ and $\sum_{m=1}^{\ell}$ are identically zero when $\ell = 0$. Observe that $\vec{F}_\ell$ and $\vec{G}_\ell$ are expressed in terms of the incident field and preceding terms in the series for $\vec{E}$ and $\vec{H}$, i.e., $\vec{E}_o, \vec{E}_1, \ldots \vec{E}_{\ell-1}, \vec{H}_o, \vec{H}_1, \ldots \vec{H}_{\ell-1}$. Thus if we consider the problem of finding $\vec{E}_\ell$ and $\vec{H}_\ell$, assuming that the preceding terms have already been determined, then $\vec{F}_\ell$ and $\vec{G}_\ell$ are known functions.

The approach, ours as well as Stevenson's, is to show that the unknown terms in (4.4a) and (4.4b) are gradients of exterior potential functions which may be determined as solutions of standard potential problems. Stevenson went to considerable effort and complication to formulate these problems. The method of this section, though still complicated, hopefully represents a simplification. In any

event, the present procedure for finding E_ℓ and H_ℓ, or at least defining them in terms of solutions of potential problems is based on the integral relation (4.4a) and (4.4b) and does not require, as Stevenson does, first finding particular solutions of Maxwell's equations.

Consider first the task of determining $\vec{E}_\ell(\vec{r})$. We observe, and this is the essence of the approach, that the unknown term on the right hand side of equation (4.4a) is itself the gradient of an exterior potential function, a single layer distribution of density $\hat{n} \cdot \vec{E}_\ell$. It is possible to formulate a boundary value problem for this term as follows. Let ϕ_ℓ denote the unknown potential, i.e.

$$\phi_\ell = -\frac{1}{4\pi} \int_B \frac{\hat{n} \cdot \vec{E}_\ell}{R} \, dB \qquad (4.6)$$

Then

$$\vec{E}_\ell = \vec{F}_\ell + \nabla\phi_\ell \qquad (4.7)$$

where $\vec{F}_\ell$ is known and

$$\nabla^2\phi_\ell = 0 \qquad\qquad \vec{r} \text{ in } V$$

ϕ_ℓ regular at infinity in the sense of Kellog

$$\hat{n} \times \nabla\phi_\ell \Big|_{\vec{r}=\vec{r}_B} = -\hat{n} \times (\vec{E}_\ell^{inc} + \vec{F}_\ell) \Big|_{\vec{r}=\vec{r}_B}$$

This is not quite a standard Dirichlet potential problem in that the boundary condition as given may be shown to specify the function ϕ_ℓ on the boundary to within a constant. That is, specifying $n \times \nabla\phi_\ell$ on B is equivalent to specifying $\phi_\ell + c$ on B where c is constant but unknown. This constant is evaluated by solving the potential problem with the ambiguous boundary condition, constructing the corresponding $\vec{E}_\ell$ with equation (4.7) and then imposing the requirement

$$\int_B \hat{n} \cdot \vec{E}_\ell \, dB = 0 \tag{4.3}$$

The procedure for finding $\vec{E}_\ell$ is thus seen to be reasonably straightforward once we observe that the unknown part of $\vec{E}_\ell$ is the gradient of an exterior potential. This observation spares us much of the complication of Stevenson's approach.

The determination of $\vec{H}_\ell$ requires more work since it is not obvious that the unknown term on the right hand side of (4.4b) is the gradient of an exterior potential, except when $\ell = 0$. In fact it may be shown that when $\ell \neq 0$, this term is definitely not the gradient of an exterior potential function. Nevertheless it is possible to retain some of the simplicity inherent in the determination of $\vec{E}_\ell$ by adding a known function to the unknown term such that the sum is the gradient of an exterior potential function. The determination of the function we must add again requires the solution of a potential problem.

Thus we introduce a function $\vec{g}_\ell$ as yet unspecified, into equation (4.4b), obtaining

$$\vec{H}_\ell(\vec{r}) = \vec{G}_\ell(\vec{r}) - \vec{g}_\ell(\vec{r}) + \frac{1}{4\pi} \nabla \times \int_B \frac{\hat{n} \times \vec{H}_\ell}{R} \, dB + \vec{g}_\ell(\vec{r}) \ . \tag{4.9}$$

It is well known and/or easily verified that a condition sufficient to guarantee that a vector be the gradient of a scalar is that the curl of the vector vanish, i.e.

$$\nabla \times \vec{A} = 0 \implies \vec{A} = \nabla \psi \ . \tag{4.10}$$

Thus a condition sufficient to guarantee that we may writ

$$\frac{1}{4\pi} \nabla \times \int_B \frac{\hat{n} \times \vec{H}_\ell}{R} \, dB + \vec{g}_\ell(\vec{r}) = \nabla \psi_\ell \tag{4.11}$$

is

$$\frac{1}{4\pi} \nabla \times \nabla \times \int_B \frac{\hat{n} \times \vec{H}_\ell}{R} dB + \nabla \times \vec{g}_\ell = 0 \qquad (4.12)$$

Since $\vec{g}_\ell$ is as yet unspecified, we use (4.12) as an equation for $\vec{g}_\ell$ and seek a particular solution in terms of known functions, i.e., terms not involving $\vec{H}_\ell$ or $\vec{E}_\ell$. Since $\vec{H}_\ell$ appears, we must first put (4.12) in suitable form. First we use the vector identity $\nabla \times \nabla \times = \nabla(\nabla \cdot - \nabla^2$ together with the fact that for $\vec{r}$ in V,

$$\nabla^2 \frac{1}{R(\vec{r}, \vec{r}_B)} = 0$$

to rewrite (4.12) as

$$\nabla \times \vec{g}_\ell(\vec{r}) = -\frac{1}{4\pi} \nabla \nabla \cdot \int_B \frac{\hat{n} \times \vec{H}_\ell}{R} dB. \qquad (4.13)$$

Recall that R is symmetric in $\vec{r}$ and $\vec{r}_B$ (eqn. 2.1) and $\nabla \frac{1}{R} = -\nabla_B \frac{1}{R}$, where ∇ operates on $\vec{r}$ and ∇_B on $\vec{r}_B$, hence (4.13) may be written

$$\nabla \times \vec{g}_\ell(\vec{r}) = \frac{1}{4\pi} \nabla \int_B \nabla_B \frac{1}{R} \cdot \hat{n} \times \vec{H}_\ell dB, \qquad (4.14)$$

or, on employing the properties of the scalar triple product,

$$\nabla \times \vec{g}_\ell(\vec{r}) = -\frac{1}{4\pi} \nabla \int_B \hat{n} \cdot \nabla_B \frac{1}{R} \times \vec{H}_\ell dB. \qquad (4.15)$$

This we rewrite as

$$\nabla \times \vec{g}_\ell(\vec{r}) = -\frac{1}{4\pi} \nabla \int_B \hat{n} \cdot (\nabla_B \times \frac{\vec{H}_\ell}{R} - \frac{1}{R} \nabla_B \times \vec{H}_\ell) dB \qquad (4.16)$$

and, since Stokes' theorem implies

$$\int_B \hat{n} \cdot \nabla_B x \frac{\vec{H}_\ell}{R} \, dB = 0, \tag{4.17}$$

we have

$$\nabla x \vec{g}_\ell(\vec{r}) = \frac{1}{4\pi} \nabla \int_B \frac{\hat{n}}{R} \cdot \nabla_B x \vec{H}_\ell \, dB . \tag{4.18}$$

But Maxwell's equations (3.10) imply that

$$\nabla_B x \vec{H}_\ell(\vec{r}_B) = -\vec{E}_{\ell-1}(\vec{r}_B) \qquad \ell > 0 \tag{4.19}$$

$$= 0 \qquad \ell = 0$$

hence we have, finally,

$$\nabla x \vec{g}_\ell(\vec{r}) = -\frac{1}{4\pi} \nabla \int_B \frac{\hat{n} \cdot \vec{E}_{\ell-1}}{R} \, dB , \qquad \ell > 0 \tag{4.20}$$

$$= 0 , \qquad \ell = 0 .$$

We have thus succeeded in rewriting (4.12) in terms of known functions since we have assumed that $\vec{E}_{\ell-1}$ is known. Now we want a particular solution of (4.20). Clearly when $\ell = 0$, $\vec{g}_0 = 0$ is a solution. When $\ell > 0$, we employ Stevenson's method for producing particular solutions of the equation $\nabla x \vec{F} = \vec{f}$. First of all note that the right hand side of (4.20) is the gradient of an exterior potential function (single layer distribution). Thus, introducing the notation

$$u_\ell^e(\vec{r}) = -\frac{1}{4\pi} \int_B \frac{\hat{n} \cdot \vec{E}_{\ell-1}}{R} \, dB , \tag{4.21}$$

(4.20) may be written

$$\nabla \times \vec{g}_\ell(\vec{r}) = \nabla u_\ell^e \qquad . \qquad \vec{r} \in V \qquad (4.22)$$

Stevenson has shown that necessary and sufficient conditions for (4.22) to have a solution are $\left[(3.18), (3.19)\right]$

$$\nabla \cdot \nabla u_\ell^e = 0 \qquad\qquad r \in V \qquad (4.23)$$

$$\int_B \frac{\partial u_\ell^e}{\partial n} \, dB = 0 \qquad\qquad (4.24)$$

The first condition, (4.23), is clearly satisfied since, as noted, u_ℓ^e is a potential function. To show that (4.24) is also satisfied we use Gauss' theorem to write

$$\int_B \frac{\partial u_\ell^e}{\partial n} \, dB = \int_B \hat{n} \cdot \nabla u_\ell^e \, dB = -\int_V \nabla^2 u_\ell^e \, dv + \int_{B_\infty} \hat{r} \cdot \nabla u_\ell^e \, dB \qquad (4.25)$$

where B_∞ denotes a large sphere whose radius approaches infinity. The volume integral term vanishes by virtue of (4.23) and the surface integral over B_∞ will also vanish if

$$\nabla u_\ell^e = o(1/r^2) \qquad\qquad \text{as } r \to \infty . \qquad (4.26)$$

That (4.26) is satisfied may be seen by examining the structure of u_ℓ^e exhibited in the defining equation (4.21). Thus

$$\nabla u_\ell^e(\vec{r}) = -\frac{1}{4\pi} \nabla \int_B \frac{\hat{n} \cdot \vec{E}_{\ell-1}(\vec{r}_B)}{R(\vec{r}, \vec{r}_B)} \, dB \qquad (4.27)$$

or, for $r > \max r_B$,

$$\nabla u_\ell^e(\vec{r}) = -\frac{1}{4\pi} \nabla \sum_{m=0}^{\infty} \int_B \hat{n} \cdot \vec{E}_{\ell-1}(\vec{r}_B) \frac{r_B^m}{r^{m+1}} P_m(\cos\gamma) \, dB \tag{4.28}$$

where $\cos\gamma = \cos\theta\cos\theta_B + \sin\theta\sin\theta_B\cos(\phi - \phi_B)$ and P_m is a Legendre polynomial of order m.

The $m = 0$ term vanishes by virtue of (4.3),

$$\int_B \hat{n} \cdot \vec{E}_\ell \, dB = 0 \quad , \qquad \ell = 0, 1, 2, \ldots,$$

hence

$$\nabla u_\ell^e(\vec{r}) = -\frac{1}{4\pi} \nabla \sum_{m=1}^{\infty} \frac{1}{r^{m+1}} \int_B \hat{n} \cdot \vec{E}_{\ell-1}(\vec{r}_B) r_B^m P_m(\cos\gamma) \, dB \tag{4.29}$$

From (4.29) it is clear that

$$\nabla u_\ell^e(\vec{r}) = O(1/r^3) \quad \text{or} \quad o(1/r^2) \quad \text{as } r \to \infty \tag{4.30}$$

hence (4.26) holds which in turn means that (4.24) is valid. Thus we have established that equation (4.22) has a solution. Furthermore we have shown, in the process, that the right hand side of (4.22) is $O(1/r^3)$ at infinity. Now we use Stevenson's solution to this problem.

We define an interior potential function, $u_\ell^i(\vec{r})$, $\vec{r}$ interior to B, as follows.

$$\nabla^2 u_\ell^i(\vec{r}) = 0 \qquad \qquad \vec{r} \text{ interior to B} \tag{4.31}$$

$$\hat{n} \cdot \nabla u_\ell^i(\vec{r})\Big|_{\vec{r}=\vec{r}_B} = \hat{n} \cdot \nabla u_\ell^e(\vec{r})\Big|_{\vec{r}=\vec{r}_B} \tag{4.32}$$

This is a standard interior Neumann problem for $u_\ell^i(\vec{r})$ and has a solution provided that

$$\int_B \hat{n} \cdot \nabla u_\ell^i \, dB = 0 \tag{4.33}$$

but this is satisfied by virtue of (4.24) and the boundary condition (4.32). Recall that $\hat{n}$ above is always directed from B into V, the exterior of B.

Now, according to Stevenson a particular solution of the equation (4.22) is given by

$$\vec{g}_\ell(\vec{r}) = \frac{i}{4\pi} \, \nabla \times \left\{ \int_V \frac{\nabla_v u_\ell^e(\vec{r}_v)}{R} \, dv + \int_{V_i} \frac{\nabla_v u_\ell^i(\vec{r}_v)}{R} \, dv \right\} \tag{4.34}$$

where V_i is the interior of B, and V the exterior.

To demonstrate that (4.34) is indeed a solution of (4.22) is a relatively simple calculation. Again using the identity $\nabla \times \nabla \times = \nabla(\nabla \cdot - \nabla^2$ we have

$$\nabla \times \vec{g}_\ell(\vec{r}) = \frac{1}{4\pi} \, \nabla \left(\nabla \cdot \int_V \frac{\nabla_v u_\ell^e(\vec{r}_v)}{R} \, dv + \nabla \cdot \int_{V_i} \frac{\nabla_v u_\ell^i(\vec{r}_v)}{R} \, dv \right)$$

$$- \frac{1}{4\pi} \nabla^2 \left(\int_V \frac{\nabla_v u_\ell^e(\vec{r}_v)}{R} \, dv + \int_{V_i} \frac{\nabla_v u_\ell^i(\vec{r}_v)}{R} \, dv \right) \quad , \quad r \in V \tag{4.35}$$

but

$$\nabla^2 \left(- \frac{1}{4\pi R(\vec{r} - \vec{r}_v)} \right) = \delta(|\vec{r} - \vec{r}_v|) \tag{4.36}$$

therefore, for $\vec{r}$ in V,

$$\nabla \times \vec{g}_\ell(\vec{r}) = \nabla u_\ell^e(\vec{r}) + \frac{1}{4\pi} \, \nabla \left(\nabla \cdot \int_V \frac{\nabla_v u_\ell^e(\vec{r}_v)}{R} \, dv + \nabla \cdot \int_{V_i} \frac{\nabla_v u_\ell^i(\vec{r}_v)}{R} \, dv \right) . \tag{4.37}$$

Now using the facts that $\nabla \frac{1}{R} = -\nabla_v \frac{1}{R}$ and u_ℓ^e and u_ℓ^i are both potential functions we obtain

$$\nabla \times \vec{g}_\ell(\vec{r}) = \nabla u_\ell^e(\vec{r}) - \frac{1}{4\pi} \nabla \int_V \nabla_v \cdot \frac{\nabla_v u_\ell^e(\vec{r}_v)}{R} \, dv - \frac{1}{4\pi} \nabla \int_{V_i} \nabla_v \cdot \frac{\nabla_v u_\ell^i(\vec{r}_v)}{R} \, dv.$$

$$(4.38)$$

Now we use Gauss' theorem, taking care of the signs of the normals ($\hat{n}$ on B is always directed into V, the exterior) to obtain

$$\nabla \times \vec{g}_\ell(\vec{r}) = \nabla u_\ell^e(\vec{r}) + \frac{1}{4\pi} \nabla \int_B \hat{n} \cdot \left[\nabla_B u_\ell^e(\vec{r}_B) - \nabla_B u_\ell^i(\vec{r}_B)\right] dB .$$

$$(4.39)$$

Actually there is another surface integral term over a large sphere at infinity but this vanishes by virtue of (4.30). The integral in (4.39) vanishes because of the boundary condition (4.32) thus verifying that

$$\nabla \times \vec{g}_\ell(\vec{r}) = \nabla u_\ell^e(\vec{r}) \quad , \qquad \vec{r} \in V .$$

We may cast $\vec{g}_\ell(\vec{r})$ in slightly more convenient form as follows. Again using the fact that $\nabla \frac{1}{R} = -\nabla_v \frac{1}{R}$, (4.34) becomes

$$\vec{g}_\ell(\vec{r}) = -\frac{1}{4\pi} \int_V \nabla_v \frac{1}{R} \times \nabla_v u_\ell^e(\vec{r}_v) \, dv - \frac{1}{4\pi} \int_{V_i} \nabla_v \frac{1}{R} \times \nabla_v u_\ell^i(\vec{r}_v) \, dv , \quad (4.40)$$

or since curl of the gradient is identically zero,

$$\vec{g}_\ell(\vec{r}) = -\frac{1}{4\pi} \int_V \nabla_v \times \frac{\nabla_v u_\ell^e(\vec{r}_v)}{R} \, dv - \frac{1}{4\pi} \int_{V_i} \nabla_v \times \frac{\nabla_v u_\ell^i(\vec{r}_v)}{R} \, dv . \quad (4.41)$$

Now employing a famous, but apparently nameless theorem of vector analysis,

$$\int_V \nabla \times \vec{A}\, dv = \int_S \hat{n} \times \vec{A}\, ds \quad (S \text{ encloses } V \text{ and } \hat{n} \text{ is out of } S),$$

equation (4.41) becomes

$$\vec{g}_\ell(\vec{r}) = \frac{1}{4\pi} \int_B \frac{\hat{n} \times \left[\nabla_B u_\ell^e(\vec{r}_B) - \nabla_B u_\ell^i(\vec{r}_B)\right]}{R}\, dB \quad , \quad \ell > 0 \qquad (4.42)$$

An alternate form of (4.42) is found to be

$$\vec{g}_\ell(\vec{r}) = -\frac{1}{4\pi} \nabla \times \int_B \frac{\left[u_\ell^e(\vec{r}_B) - u_\ell^i(\vec{r}_B)\right]\hat{n}}{R}\, dB \qquad (4.43)$$

Again the behavior of ∇u_ℓ^e at infinity, (4.30), causes a similar integral over a large sphere to vanish.

In this form it is clear that the tangential components of ∇u_ℓ^e and ∇u_ℓ^i on B must be unequal if $\vec{g}_\ell$ is to be different from zero. In fact they are necessarily discontinuous. Since u_ℓ^e is a potential function regular exterior to B, u_ℓ^i is a potential function regular interior to B and their normal derivatives were defined to be continuous at B, then the tangential derivatives cannot also be continuous. If so, u_ℓ^i would be a continuation into the interior of B of u_ℓ^e. The resulting function would be a potential function regular everywhere in space and therefore would necessarily be zero. But u_ℓ^e (see eqn. 4.21) is not identically zero.

We have thus determined a particular $\vec{g}_\ell$ such that (4.12) is satisfied. This in turn guarantees that equation (4.11) holds, that is, with the $\vec{g}_\ell$ we have found we may write

$$\frac{1}{4\pi} \nabla \times \int_B \frac{\hat{n} \times \vec{H}_\ell}{R} \, dB + \vec{g}_\ell = \nabla \psi_\ell \,. \tag{4.11}$$

With equation (4.34) or (4.43) which expresses $\vec{g}_\ell$ as a curl, it follows upon taking the divergence of (4.11) that

$$\nabla^2 \psi_\ell = 0 \,.$$

On expanding $1/R$ in (4.43) it follows that

$$\vec{g}_\ell(\vec{r}) = O(1/r^2) \qquad \text{as} \qquad r \to \infty$$

Also

$$\nabla \times \int_B \frac{\hat{n} \times \vec{H}_\ell}{R} \, dB = O(1/r^2) \quad \text{as} \quad r \to \infty$$

hence ψ_ℓ is regular in the sense of Kellog. (Actually $\nabla \psi_\ell = O(1/r^2)$ does not imply completely that ψ_ℓ is regular. There may be an additive constant which would imply $|r\psi_\ell|$ is not bounded. Since we are interested in $\nabla \psi_\ell$, which removes this constant anyway, we may choose it as zero to begin with and take ψ_ℓ to be regular.)

With equation (4.9) and the boundary conditions (4.2) we may formulate a standard exterior Neumann potential problem for ψ_ℓ, namely

$$\nabla^2 \psi_\ell = 0 \qquad \vec{r} \in V$$

ψ_ℓ regular at infinity in the sense of Kellog $\tag{4.44}$

$$\hat{n} \cdot \nabla \psi_\ell \Big|_{\vec{r}=\vec{r}_B} = \frac{\partial \psi_\ell}{\partial n} \Big|_{\vec{r}=\vec{r}_B} = \left\{ -\hat{n} \cdot \vec{H}_\ell^{\text{inc}} - \hat{n} \cdot \vec{G}_\ell + \right.$$
$$\left. + \frac{1}{4\pi} \hat{n} \cdot \int_B \frac{\hat{n} \times \nabla_B(u_\ell^e - u_\ell^i)}{R} \right\} \Big|_{\vec{r}=\vec{r}_B}$$

The solution of this problem then determines $\vec{H}_\ell(\vec{r})$.

To summarize the procedure we have established:

If an electromagnetic field,

$$\vec{E}^{inc}(\vec{r}) = \sum_{\ell=0}^{\infty} \vec{E}_\ell^{inc}(\vec{r})(ik)^\ell \quad , \qquad \vec{H}^{inc}(\vec{r}) = \sum_{\ell=0}^{\infty} \vec{H}_\ell^{inc}(\vec{r})(ik)^\ell \qquad (4.45)$$

is incident on a smooth finite perfectly conducting surface B in three space then the coefficients in the low frequency expansion of the scattered field

$$\vec{E}(\vec{r}) = \sum_{\ell=0}^{\infty} \vec{E}_\ell(\vec{r})(ik)^\ell \quad , \qquad \vec{H}(\vec{r}) = \sum_{\ell=0}^{\infty} \vec{H}_\ell(\vec{r})(ik)^\ell \qquad (4.46)$$

are given by

$$\vec{E}_\ell(\vec{r}) = \vec{F}_\ell(\vec{r}) + \nabla\phi_\ell \qquad (4.47)$$

$$\vec{H}_\ell(\vec{r}) = \vec{G}_\ell(\vec{r}) - \vec{g}_\ell(\vec{r}) + \nabla\psi_\ell \qquad (4.48)$$

where

$$\vec{F}_\ell(\vec{r}) = -\frac{1}{4\pi}\nabla \times \sum_{m=0}^{\ell} \frac{1}{m!} \int_B \hat{n} \times \vec{E}_{\ell-m}^{inc} R^{m-1} dB + \frac{1}{4\pi}\sum_{m=0}^{\ell-1}\frac{1}{m!}$$

$$\cdot \int_B \hat{n} \times \vec{H}_{\ell-m-1} R^{m-1} dB - \frac{1}{4\pi}\sum_{m=1}^{\ell}\frac{1}{m!}\nabla\int_B \hat{n}\cdot\vec{E}_{\ell-m} R^{m-1} dB, \quad \ell > 0$$

$$(4.49)$$

$$\vec{F}_o(\vec{r}) = -\frac{1}{4\pi}\nabla \times \int_B \frac{\hat{n}\times\vec{E}_o^{inc}}{R} dB$$

$$\vec{G}_\ell(\vec{r}) = \frac{1}{4\pi} \nabla \times \sum_{m=1}^{\ell} \frac{1}{m!} \int_B \hat{n} \times H_{\ell-m} R^{m-1} dB + \frac{1}{4\pi} \sum_{m=0}^{\ell-1} \frac{1}{m!} \int_B \hat{n} \times \vec{E}_{\ell-m-1}^{inc} R^{m-1} dB$$

$$+ \frac{1}{4\pi} \sum_{m=0}^{\ell} \frac{1}{m!} \nabla \int_B \hat{n} \cdot \vec{H}_{\ell-m}^{inc} R^{m-1} dB, \qquad \ell > 0$$

$$(4.50)$$

$$\vec{G}_0(\vec{r}) = \frac{1}{4\pi} \nabla \int_B \frac{\hat{n} \cdot \vec{H}_o^{inc}}{R} dB$$

$$\vec{g}_\ell(\vec{r}) = \frac{1}{4\pi} \int_B \frac{\hat{n} \times \nabla_B (u_\ell^e - u_\ell^i)}{R} dB = -\frac{1}{4\pi} \nabla \times \int_B \frac{\hat{n}}{R} (u_\ell^e - u_\ell^i) dB, \qquad \ell > 0$$

$$\vec{g}_0(\vec{r}) = 0 \tag{4.51}$$

$$u_\ell^e(\vec{r}) = -\frac{1}{4\pi} \int_B \frac{\hat{n} \cdot \vec{E}_{\ell-1}}{R} dB \qquad , \qquad \ell > 0 \tag{4.52}$$

and u_ℓ^i, ϕ_ℓ, and ψ_ℓ are all solutions of standard potential problems.

$u_\ell^i(\vec{r})$ is an interior Neumann potential:

$$\left. \begin{array}{ll} \nabla^2 u_\ell^i = 0 & \vec{r} \text{ interior to B} \\[2ex] \hat{n} \cdot \nabla u_\ell^i = \hat{n} \cdot \nabla u_\ell^e & \text{on B ,} \end{array} \right\} \tag{4.53}$$

ϕ_ℓ is an exterior Dirichlet potential:

$$\left. \begin{array}{ll} \nabla^2 \phi_\ell = 0 & \vec{r} \text{ exterior to B} \\[2ex] \phi_\ell \text{ reg at } \infty & \\[2ex] \hat{n} \times \nabla \phi_\ell = -\hat{n} \times (\vec{E}_\ell^{inc} + \vec{F}_\ell) & \text{on B} \end{array} \right\} \tag{4.54}$$

(we must use $\displaystyle\int_B \hat{n} \cdot \vec{E}_\ell\, dB = 0$ to determine an arbitrary constant arising from this form of the boundary condition),

and ψ_ℓ is an exterior Neumann potential:

$$\left.\begin{aligned} \nabla^2\psi_\ell &= 0 \qquad\qquad \vec{r} \text{ exterior to B}\\[1em] \psi_\ell &\text{ reg at } \infty \\[1em] \hat{n} \cdot \nabla\psi_\ell &= -\hat{n} \cdot \vec{H}_\ell^{inc} - \hat{n} \cdot \vec{G}_\ell + \hat{n} \cdot \vec{g}_\ell \quad \text{ on B} \end{aligned}\right\} \tag{4.55}$$

$\hat{n}$ is the unit normal on B always pointing into V, the exterior of B, and R is the distance $|\vec{r} - \vec{r}_B|$ from a point $\vec{r}_B$ on the surface (the integration variables) to a field point $\vec{r}$.

We complete this section with a brief discussion of the low frequency expansion of the far field. Here we proceed exactly as in the scalar case. We incorporate the facts that, for large r,

$$\frac{e^{ikR}}{R} \sim \frac{e^{ik(r - \hat{r} \cdot \vec{r}_B)}}{r} \quad , \qquad \nabla\frac{e^{ikR}}{R} \sim ik\hat{r}\,\frac{e^{ik(r - \hat{r} \cdot \vec{r}_B)}}{r}$$

in the Stratton-Chu integral representations of the scattered field, equations (3.4a) and (3.4b), also employing the boundary conditions on the surface (3.12), obtaining

$$\vec{E}(\vec{r}) \sim \frac{e^{ikr}}{4\pi r}ik \int_B e^{-ik\hat{r} \cdot \vec{r}_B} \left[-\hat{r} \times (\hat{n} \times \vec{E}^{inc}) + \hat{n} \times \vec{H} - \hat{r}\hat{n} \cdot \vec{E}\right] dB \tag{4.56}$$

$$\vec{H}(\vec{r}) \sim \frac{e^{ikr}}{4\pi r}ik \int_B e^{-ik\hat{r} \cdot \vec{r}_B} \left[\hat{r} \times (\hat{n} \times \vec{H}) + \hat{n} \times \vec{E}^{inc} + \hat{r}\hat{n} \cdot \vec{H}^{inc}\right] dB \tag{4.57}$$

Now we expand the field quantities, equations (4.45) and (4.46) and the factor

$e^{-ik\hat{r}\cdot\vec{r}_B}$ and reorder the terms to find

$$\vec{E}(\vec{r}) \sim \frac{e^{ikr}}{4\pi r} \sum_{\ell=0}^{\infty} (ik)^{\ell+1} \sum_{m=0}^{\ell} \frac{(-1)^{\ell-m}}{(\ell-m)!} \int_B (\hat{r}\cdot\vec{r}_B)^{\ell-m} \left[-\hat{r}\times(\hat{n}\times\vec{E}_m^{inc}) \right.$$

$$\left. + \hat{n}\times\vec{H}_m - \hat{r}\,\hat{n}\cdot\vec{E}_m \right] dB \qquad (4.58)$$

$$\vec{H}(\vec{r}) \sim \frac{e^{ikr}}{4\pi r} \sum_{\ell=0}^{\infty} (ik)^{\ell+1} \sum_{m=0}^{\ell} \frac{(-1)^{\ell-m}}{(\ell-m)!} \int_B (\hat{r}\cdot\vec{r}_B)^{\ell-m} \left[\hat{r}\times(\hat{n}\times\vec{H}_m) \right.$$

$$\left. + \hat{n}\times\vec{E}_m^{inc} + \hat{r}\,\hat{n}\cdot\vec{H}_m^{inc} \right] dB \qquad (4.59)$$

The $\ell=0$ term in (4.58) and (4.59) always vanishes $\left[\text{the integrals} \int_B \hat{n}\cdot\vec{E}_m \, dB \right.$
and $\int_B \hat{n}\cdot\vec{H}_m$ are zero for all m, (equation 4.3) and, since $\vec{E}_o^{inc}$ and $\vec{H}_o$ may
both be written as gradients of potential functions, we may use a well known result of vector analysis which implies that $\int_B \hat{n}\times\nabla\phi \, dB \equiv 0$, for B closed and any
ϕ to see that $\int_B \hat{n}\times\vec{H}_o \, dB$ and $\left. \int_B \hat{n}\times\vec{E}_o^{inc} \, dB \right.$ also vanish. $\Big]$. Therefore we may
rewrite (4.58) and (4.59) as

$$\vec{E}(\vec{r}) \sim -\frac{e^{ikr}}{4\pi r} \, k^2 \sum_{\ell=0}^{\infty} (ik)^{\ell} \sum_{m=0}^{\ell+1} \frac{(-1)^{\ell+1-m}}{(\ell+1-m)!} \int_B (\hat{r}\cdot\vec{r}_B)^{\ell+1-m}$$

$$\cdot \left[-\hat{r}\times(\hat{n}\times\vec{E}_m^{inc}) + \hat{n}\times\vec{H}_m - \hat{r}\,\hat{n}\cdot\vec{E}_m \right] dB$$

$$(4.60)$$

and

$$\vec{H}(\hat{r}) \sim -\frac{e^{ikr}}{4\pi r}k^2 \sum_{\ell=0}^{\infty}(ik)^{\ell}\sum_{m=0}^{\ell+1}\frac{(-1)^{\ell+1-m}}{(\ell+1-m)!}\int_B (r\cdot r_B)^{\ell+1-m}$$

$$\cdot\left[\hat{r}\times(\hat{n}\times\vec{H}_m) + \hat{n}\times\vec{E}_m^{inc} + \hat{r}\,\hat{n}\cdot\vec{H}_m^{inc}\right]dB \quad.$$

$$(4.61)$$

This illustrates a famous result of Rayleigh: the leading term in a low frequency expansion of the far field is proportional to k^2. Stevenson criticized this form as being inefficient since one apparently needs to determine $\ell+1$ non-vanishing near field terms in order to obtain ℓ non-vanishing far field terms. Actually this is not completely true, as a close examination of the "extra" near field terms reveals. These are the $m = \ell+1$ terms in (4.60) and (4.61), namely

$$\int_B\left[-\hat{r}\times(\hat{n}\times E_{\ell+1}^{inc}) + \hat{n}\times\vec{H}_{\ell+1} - \hat{r}\,\hat{n}\cdot\vec{E}_{\ell+1}\right]dB \qquad (4.62a)$$

and

$$\int_B\left[\hat{r}\times(\hat{n}\times\vec{H}_{\ell+1}) + \hat{n}\times\vec{E}_{\ell+1}^{inc} + \hat{r}\,\hat{n}\cdot\vec{H}_{\ell+1}^{inc}\right]dB \qquad (4.62b)$$

which we rewrite as

$$-\hat{r}\times\int_B\hat{n}\times\vec{E}_{\ell+1}^{inc}\,dB + \int_B\hat{n}\times\vec{H}_{\ell+1}\,dB - \hat{r}\int_B\hat{n}\cdot\vec{E}_{\ell+1}\,dB \qquad (4.63a)$$

and

$$\hat{r}\times\int_B\hat{n}\times\vec{H}_{\ell+1}\,dB + \int_B\hat{n}\times\vec{E}_{\ell+1}^{inc}\,dB + \hat{r}\int_B\hat{n}\cdot\vec{H}_{\ell+1}^{inc}\,dB \quad. \qquad (4.63b)$$

The terms involving the incident field are effectively known since the incident field is given. Also, from equation (3.14), $\int_B\hat{n}\cdot\vec{E}_{\ell+1}\,dB = 0$ and the only

unknown part of these "extra" terms involves

$$\int_B \hat{n} \times \vec{H}_{l+1} \, dB \, . \tag{4.64}$$

With (4.48), however, it follows that

$$\int_B \hat{n} \times \vec{H}_{l+1} \, dB = \int_B (\hat{n} \times \vec{G}_{l+1} - \hat{n} \times \vec{g}_{l+1} + \hat{n} \times \nabla \psi_{l+1}) \, dB \tag{4.65}$$

But $\int_B \hat{n} \times \nabla \psi_{l+1} \, dB = 0$, using a vector identify we have employed before, and $\vec{G}_{l+1}$ is given, (4.50), in terms of the first l near field terms. The only unknown part of this "extra" term involves $\vec{g}_{l+1}$ which does require the solution of an interior Neumann problem see equations (4.51)-(4.53). This is considerably less than requiring complete determination of E_{l+1} and H_{l+1}, but is still unsatisfactory. Repeated attempts to determine this "unknown" part without solving for $\vec{g}_{l+1}$ have so far been fruitless. The alternatives are also less than overwhelmingly desirable. Stevenson provides a generalization of Rayleigh's continuation method whereby the near field terms for large r are matched with multipoles for small k (thus defining the multipole moments) then using the far fields of the multipoles. This of course involves expanding the near field terms in spherical harmonics which may involve as much labor as solving the required interior Neumann problem. Still, in principle, Stevenson's method of continuing into the far field is preferable since it does not require the solution of another, albeit simple, problem in order to obtain the same number of terms in a low frequency expansion of the far field as are available in a low frequency expansion of the near field. The price is apparently requiring both to be represented as expansions in spherical harmonics.

V
AN EXAMPLE—SCATTERING BY A SPHERE

To illustrate the procedure derived in the previous section, we consider the problem of scattering of a linearly polarized plane wave by a sphere. The incident field is taken to propagate down the z-axis, with $\vec{E}^{inc}$ along the x-direction (see Fig. 2), i.e.

$$\vec{E}^{inc} = \hat{i}_x e^{-ikz} = \sum_{\ell=0}^{\infty} (ik)^{\ell} \vec{E}^{inc}_{\ell} \quad , \qquad \vec{E}^{inc}_{\ell} = \frac{(-z)^{\ell}}{\ell!} \hat{i}_x$$

$$\vec{H}^{inc} = -\hat{i}_y e^{-ikz} = \sum_{\ell=0}^{\infty} (ik)^{\ell} \vec{H}^{inc}_{\ell} \quad , \qquad \vec{H}^{inc}_{\ell} = -\frac{(-z)^{\ell}}{\ell!} \hat{i}_y$$
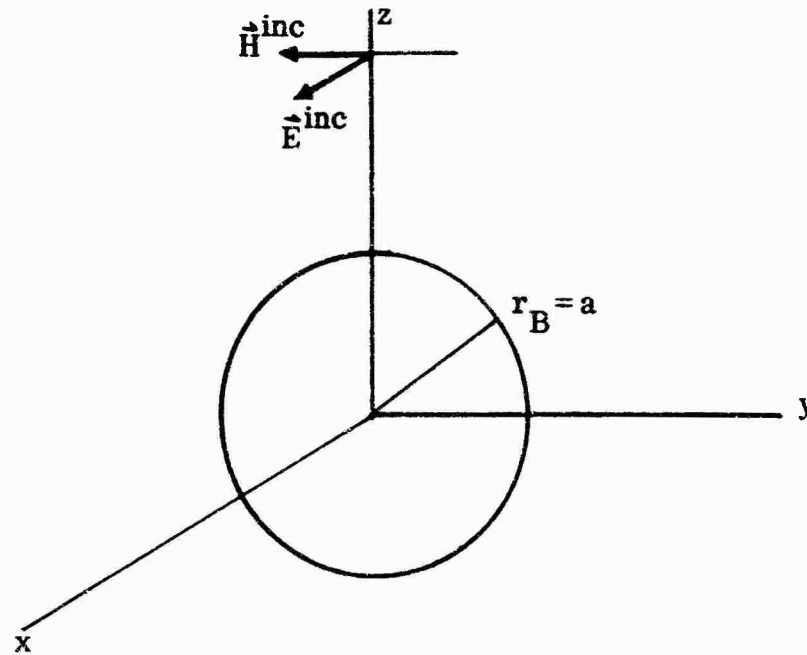
$$(5.1)$$

Figure 2

We shall proceed to calculate the first two terms in the series for the scattered field,

$$\vec{E} = \sum_{l=0}^{\infty} (ik)^l \vec{E}_l \ , \qquad\qquad \vec{H} = \sum_{l=0}^{\infty} (ik)^l \vec{H}_l \ , \qquad (5.2)$$

by straightforward application of equations (4.46) - (4.55).

5.1 Zeroth Order Terms

From (5.1) we see that

$$\vec{E}_o^{inc} = \hat{i}_x \ . \qquad (5.3)$$

The scattered electric field to this order is (4.47)

$$\vec{E}_o = \vec{F}_o + \nabla \phi_o \qquad (5.4)$$

and (Eq. 4.49)

$$\vec{F}_o = -\frac{1}{4\pi} \nabla x \int_B \frac{\hat{n} \, x \, \hat{i}_x}{R} \ dB \ . \qquad (5.5)$$

Equation (5.5), written in its entirety is,

$$\vec{F}_o(\vec{r}) = -\frac{1}{4\pi} \nabla x \int_0^{2\pi} d\phi_B \int_0^{\pi} d\theta_B \, a^2 \sin\theta_B \ \cdot$$

$$\qquad (5.6)$$

$$\cdot \frac{\hat{r}_B \, x \, \hat{i}_x}{\sqrt{r^2 + a^2 - 2ar\left[\cos\theta\cos\theta_B + \sin\theta\sin\theta_B\cos(\phi - \phi_B)\right]}}$$

where the unit normal is $\hat{r}_B = \hat{i}_x \sin\theta_B \cos\phi_B + \hat{i}_y \sin\theta_B \sin\phi_B + \hat{i}_z \cos\theta_B$.

The integration is carried out using the well known expansion of $1/R$ in spherical harmonics and we find (using a mixture of rectangular and spherical unit vectors)

$$\vec{F}_o(\vec{r}) = a^3 \frac{x}{r^3} \hat{r} - \frac{1}{3} \frac{a^3}{r^3} \hat{i}_x \tag{5.7}$$

Now we use (4.54) to find ϕ_o. The boundary condition

$$\hat{n} \times \nabla \phi_o = -\hat{n} \times (\vec{E}_o^{inc} + \vec{F}_o) \quad , \qquad r = a \tag{5.8}$$

is seen to imply that

$$\phi_o \Big|_{r=a} = -\frac{2a}{3} \sin\theta \cos\phi + c \ . \tag{5.9}$$

The exterior potential function taking on this boundary value is found to be (write ϕ_o as a series expansion $\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} P_n(\cos\gamma)$ whose unknown coefficients are determined using the boundary condition),

$$\phi_o = -\frac{2}{3} \frac{a^3}{r^2} \sin\theta \cos\phi + \frac{ac}{r} \tag{5.10}$$

Substituting (5.7) and (5.10) in (5.4) we find

$$\vec{E}_o = 3 \frac{a^3 x}{r^4} \hat{r} - \frac{a^3}{r^3} \hat{i}_x - \frac{ac}{r^2} \hat{r} \quad . \tag{5.11}$$

The auxilliary condition

$$\int_B \hat{n} \cdot \vec{E}_o = 0 \tag{5.12}$$

implies, with (5.11), that

$$c = 0 \ . \tag{5.13}$$

To find $\vec{H}_o$ we see (Eqs. 4.48, 4.51) that

$$\vec{H}_o = \vec{G}_o + \nabla \psi_o .$$ (5.14)

From (4.50)

$$\vec{G}_o = \frac{1}{4\pi} \nabla \oint_B \frac{\hat{n} \cdot \vec{H}_o^{inc}}{R} dB$$ (5.15)

and

$$\vec{H}_o^{inc} = -\hat{i}_y$$ (5.16)

from which

$$\vec{G}_o = -\frac{1}{3} \nabla \left(\frac{a^3 y}{r^3} \right)$$ (5.17)

ψ_o (Eq. 4.55) is an exterior potential function with boundary values

$$\frac{\partial \psi_o}{\partial r} \bigg|_{r=a} = -\hat{r} \cdot \vec{H}_o^{inc} - \hat{r} \cdot \vec{G}_o = \hat{r} \cdot \hat{i}_y + \frac{1}{3} \frac{\partial}{\partial r} \left(\frac{a^3}{r^3} y \right)$$ (5.18)

$$= \frac{1}{3} \sin\theta \sin\phi.$$

Such a function is easily seen to be

$$\psi_o = -\frac{1}{6} \frac{a^3}{r^2} \sin\theta \sin\phi$$ (5.19)

which, with (5.14) and (5.17) leads to

$$\vec{H}_o = -\frac{1}{2} \nabla \left(\frac{a^3 y}{r^3} \right) = -\frac{a^3 \hat{i}_y}{2r^3} + \frac{3}{2} \frac{a^3 y}{r^4} \hat{r}$$ (5.20)

The zeroth order results may be rewritten entirely in terms of spherical unit vectors as

$$\vec{E}_o = \frac{\hat{r}2a^2}{r^3} \sin\theta\cos\phi - \hat{\theta}\frac{a^3}{r^3}\cos\theta\cos\phi + \hat{\phi}\frac{a^3}{r^3}\sin\phi \qquad (5.21)$$

$$\vec{H}_o = \frac{\hat{r}a^3}{r^3}\sin\theta\sin\phi - \hat{\theta}\frac{a^3}{2r^3}\cos\theta\sin\phi - \hat{\phi}\frac{a^3}{2r^3}\cos\phi \;. \qquad (5.22)$$

5.2 First Order Terms

The next terms are found using these results, again following the procedure of the preceding section. It is to be noted that even at this stage, the calculations become tedious. With (4.49) we see that

$$\vec{F}_1 = -\frac{1}{4\pi}\nabla x \int_B \frac{\hat{n} x \vec{E}_1^{inc}}{R} \, dB - \frac{1}{4\pi}\nabla x \int_B \hat{n} x \vec{E}_o^{inc} \, dB + \frac{1}{4\pi}\int_B \frac{\hat{n} x \vec{H}_o}{R} \, dB$$

$$- \frac{1}{4\pi}\nabla \int_B \hat{n} \cdot \vec{E}_o \, dB \qquad (5.23)$$

All terms are well defined, $\vec{E}_o$ and $\vec{H}_o$ in (5.21) and (5.22) above and $\vec{E}_o^{inc}$ and $\vec{E}_1^{inc}$ in (5.1), namely

$$\vec{E}_o^{inc} = \hat{i}_x \quad , \qquad\qquad \vec{E}_1^{inc} = -z\hat{i}_x \quad . \qquad (5.24)$$

Carrying out the indicated integrations we find

$$\vec{F}_1 = -\frac{a^5 xz}{r^6}\hat{r} - \frac{a^3}{2r^2}\hat{r} x \hat{i}_y + \frac{a^5}{5r^5}(z\hat{i}_x + x\hat{i}_z) \qquad (5.25)$$

Now we use (4.54) to find ϕ_1. The boundary condition

$$\hat{n} x \nabla\phi_1 = -\hat{n} x (\vec{E}_1^{inc} + \vec{F}_1) \quad , \qquad r = a$$

is seen to imply

$$\phi_1\Big|_{r=a} = \frac{3a^2}{20}\cos\phi\sin2\theta+c = -\frac{a^2}{10}P_2^1(\cos\theta)\cos\phi+c \qquad (5.26)$$

Here we employ the definition of the associated Legendre functions given by Magnus and Oberhettinger (1949). The exterior potential function taking on these boundary values is found to be

$$\phi_1 = -\frac{a^5}{10r^3}P_2^1(\cos\theta)\cos\phi+\frac{ac}{r} \qquad . \qquad (5.27)$$

Forming $\vec{E}_1$ with (4.47), and applying the auxilliary condition $\displaystyle\int_B \hat{n}\cdot\vec{E}_1\,dB = 0$ which implies $c = 0$, we find

$$\vec{E}_1 = -\frac{a^3}{2r^2}\hat{r}\times\hat{i}_y+\frac{1}{2}\nabla\left(\frac{a^5\,xz}{r^5}\right) \qquad (5.28)$$

Proceeding to the determination of $\vec{H}_1$ we see that (Eq. 4.48)

$$\vec{H}_1 = \vec{G}_1-\vec{g}_1+\nabla\psi_1 \qquad . \qquad (5.29)$$

With (4.50)

$$\vec{G}_1 = \frac{1}{4\pi}\nabla\times\int_B\hat{r}_B\times\vec{H}_o\,dB + \frac{1}{4\pi}\int\frac{\hat{r}_B\times\vec{E}_o^{inc}}{R}\,dB + \frac{1}{4\pi}\nabla\int_B\frac{\hat{r}_B\cdot\vec{H}_1^{inc}}{R}\,dB$$

$$+ \frac{1}{4\pi}\nabla\int_B\hat{r}_B\cdot\vec{H}_o^{inc}\,dB \qquad . \qquad (5.30)$$

All the quantities involved have already been defined in (5.1) and (5.22). Carrying out the integration yields

$$\vec{G}_1 = \frac{a^3}{3r^2}\hat{r}\times\hat{i}_x+\nabla\left(\frac{a^5}{5r^5}yz\right) \qquad . \qquad (5.31)$$

To determine g_1 we must first find u_1^e and u_1^i. From (4.52)

$$u_1^e = -\frac{1}{4\pi} \int_B \frac{\hat{n} \cdot \vec{E}_o}{R} \, dB \tag{5.32}$$

which, with (5.21), may be evaluated as

$$u_1^e = \frac{2}{3} \frac{a^3}{r^2} P_1^1(\cos\theta) \cos\phi \tag{5.33}$$

Following (4.53) we determine the interior potential function whose normal derivative matches that of u_1^e on the boundary. Here we assume a series of the form $\sum_{n=0}^{\infty} a_n r^n P_n(\cos\gamma)$ and determine the a_n using the boundary condition. In the present case, this is easily seen to be

$$u_1^i = -\frac{4}{3} r P_1^1(\cos\theta) \cos\phi \ . \tag{5.34}$$

With (4.51) we see that

$$\vec{g}_1 = \frac{1}{4\pi} \int_B \frac{\hat{n} \times \nabla_B (u_1^e - u_1^i)}{R} \, dB \tag{5.35}$$

and using the expressions (5.33) and (5.34) this may be found explicitly as

$$\vec{g}_1 = -\frac{2}{3} \frac{a^3}{r^2} \hat{r} \times \hat{i}_x \ . \tag{5.36}$$

Now we proceed using (4.55) to find an exterior potential function, ψ_1, satisfying the boundary condition

$$\left. \frac{\partial \psi_1}{\partial n} \right|_{r=a} = -\hat{n} \cdot \vec{H}_1^{inc} - \hat{n} \cdot \vec{G}_1 + \hat{n} \cdot \vec{g}_1 \ . \tag{5.37}$$

With (5.1), (5.31) and (5.36), this boundary condition becomes

$$\frac{\partial \psi_1}{\partial r}\bigg|_{r=a} = -\frac{2}{3}\, a \sin\theta \cos\theta \sin\phi = \frac{2a}{15}\, P_2^1(\cos\theta) \sin\phi. \qquad (5.38)$$

and the solution is found to be

$$\psi_1 = -\frac{2a^5}{45r^3}\, P_2^1(\cos\theta) \sin\phi = \frac{2a^5 yz}{15 r^5}\, . \qquad (5.39)$$

Substituting (5.31), (5.35) and (5.39) in (5.29) we obtain

$$\vec{H}_1 = \frac{a^3}{r^2}\,\hat{r} \times \hat{i}_x + \frac{1}{3}\,\nabla\left(\frac{a^5 yz}{r^5}\right) \, . \qquad (5.40)$$

The first order terms may be written entirely in spherical coordinates as

$$\vec{E}_1 = -\frac{3}{2}\,\frac{a^5}{r^4}\, \sin\theta \cos\theta \cos\phi\,\hat{r} + \left(\frac{a^5}{2r^4}\cos 2\theta + \frac{a^3}{2r^2}\right) \cos\phi\,\hat{\theta}$$

$$-\cos\theta \sin\phi \left(\frac{a^3}{2r^2} + \frac{a^5}{2r^4}\right)\hat{\phi} \qquad (5.41)$$

$$\vec{H}_1 = -\frac{a^5}{r^4}\, \sin\theta \cos\theta \sin\phi\,\hat{r} + \left(\frac{a^5}{3r^4}\cos 2\theta + \frac{a^3}{r^2}\right) \sin\phi\,\hat{\theta}$$

$$+\cos\theta \cos\phi \left(\frac{a^3}{r^2} + \frac{a^5}{3r^4}\right)\hat{\phi} \qquad (5.42)$$

These results for the first two terms in the low frequency expansion may be shown to be in complete agreement with comparable expressions derived from the standard Mie series.

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
The University of Michigan	Unclassified
	2b. GROUP

3. REPORT TITLE

LOW FREQUENCY SOLUTION OF THREE-DIMENSIONAL SCATTERING PROBLEMS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(Last name, first name, initial)*

Kleinman, Ralph E.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1965	51	16

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
AF 19(628)-4328	7133-4-T
b. PROJECT NO. 5635	
c. 563502	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	AFCRL-65-639

10. AVAILABILITY/LIMITATION NOTICES

Qualified requestors may obtain copies of this report directly from DDC.
Other qualified users shall request through CFSTI.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Hq., AFCRL, OAR (CRD) United States Air Force L.G. Hanscom Field, Bedford, Mass.

13. ABSTRACT

A deficiency is pointed out in Stevenson's method of reducing the solution of electromagnetic scattering problems to a succession of standard potential problems whose solutions determine terms in the low frequency expansion of the scattered field. An alternate approach is presented, for perfectly conducting scatterers, which not only removes the difficulty but also is simpler and more explicit than Stevenson's method. The details of the analogous, though simpler, scalar scattering problems are also presented.

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